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Stochastic dominance on optimal portfolio with one risk−less and two risky assets

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Abstract

The paper provides restrictions on the investor's utility function which are sufficient for a dominating shift no decrease in the investment in the respective asset if there are one risk free asset and two risky assets in the portfolio. The analysis is then confined to portfolio in which the distributions of assets differ by a first−degree−stochastic dominance shift.

Citation: Nguema, Jean Fernand, (2005) "Stochastic dominance on optimal portfolio with one risk−less and two risky assets." *Economics Bulletin,* Vol. 7, No. 7 pp. 1−7
Submitted: June 29, 2005. **Accepted:** July 25, 2005.
URL: http://www.economicsbulletin.com/2005/volume7/EB−05G10007A.pdf

1. Introduction

The question of whether risk-averse investors should or should not choose diversified portfolio has been analysed quite extensively in the literature; for exemple Fishburn and Porter (1976), Kira and Ziemba (1980), Meyer and Orminston (1989) and recently Meyer and Orminston (1994) extend and generalize this literature by considering portfolios containing more than one risky asset. The first research assumes that the risky returns are independently distributed and the last ones, focuses on removing this independence restriction, allowing the risky returns to be stochastically dependent. All of these studies were limited to one decision variable, so they were not able to analyze the effects of generalized FDS on optimal portfolio with three assets or more. We extend these earlier works by considering portfolios with one risk free asset and two risky assets, and we analyze the effects of shift in the sense of first-degree stochastic dominance (FDS).

The analysis of comparative static's for portfolios with two or more asset is rather difficult. Following Hart (1975), in order to derive the desired effects of changes in the investor's wealth, the utility function must possess a particular type of separation property, and, as it turn out, not many classes of utility functions satisfy this property. For the case of portfolio with one risk free asset and two risky assets we were able to determine the effects of shifts mentioned above.

Our paper provides very simple conditions on the utility function which are necessary and sufficient for a dominating shift in the distribution of asset i not to cause a decrease in the investment in that asset. Two implications of independence restriction play an important role in our analysis. First, independence[1] allows one risky return to be altered without changing the marginal or the conditional cumulative distribution functions (CDF) for the others. Second, when the return to a risky asset is independent[2] of other risky returns, the marginal and each of the conditional CDF's for that return can be changed in exactly the same way.

The paper is organized as follows. In the next section, the notation and assumptions of one risk free asset and two risky assets portfolio decision model are presented and one main comparative static result is viewed. Following this, comparative static issues that only arise with stochastic independence are discussed in section 3. In section 4, for First stochastic dominance change, we determine the conditions on the decision maker's preferences that are necessary and sufficient for the change to cause an increase in the proportion of wealth invested in the asset whose return is altered. Finally, concluding remarks appear in the end.

2. The model and assumptions

We consider a risk-averse individual endowed with non-random initial wealth who allocates his wealth normalized to one between one risk free asset (with return x_0) and two risky assets with returns $\tilde{y}$ and $\tilde{z}$. The portfolio share of risky assets $\tilde{y}$ and $\tilde{z}$ are a_1 and a_2, respectively. The individual's end of period wealth $\tilde{W}$ is then equal to

$$\tilde{W}(a_i, \tilde{y}, \tilde{z}) = 1 + x_0 + a_1(\tilde{y} - x_0) + a_2(\tilde{z} - x_0) \tag{1}$$

Realizations of $\tilde{W}$ depend on realizations of $\tilde{y}$, $\tilde{z}$ and the selected values for a_1 and a_2. The random returns $\tilde{y}$ and $\tilde{z}$ are assume to take values in the interval $[0, 1]$.

The joint cumulative distribution function (CDF) for $\tilde{y}$ and $\tilde{z}$ is denoted $H(y, z)$. The conditional and marginal CDF's for $\tilde{y}$ are denoted $F(y|z)$ and $F(y)$, respectively, and for

[1] Hadar and Seo (1990) make this their ceteris paribus assumption.

[2] Hadar and Seo (1990) assume this when they require the risky returns to be independent both before and after one return is altered.

$\tilde{z}$ there are $G(z|y)$ and $G(z)$. If $\tilde{y}$ and $\tilde{z}$ are independently distributed then $F(y|z)=F(y)$ for all z, $G(z|y)=G(z)$ for all y, and $H(y, z)=F(y)\cdot G(z)$.

The decision maker is assumed to choose (a_1, a_2) to maximize the expected utility (EU) from the terminal wealth taking random variable $\tilde{y}$ and $\tilde{z}$ as given. Formally, (a_1, a_2) is selected to maximize EU, i.e. optimal portfolio solves the following program (P):

$$\max_{a_1, a_2} \int_0^1\int_0^1 u(W(a_i, \tilde{y}, \tilde{z}))d^2H(y, z) \tag{2}$$

To simplify notation, the symbol $d^2H(y,z)$ is used to denoted $\left[\frac{\partial^2 H(y, z)}{\partial y \cdot \partial z}\right]dydz$ and u is the von Neumann Morgenstern utility function which is assumed to be three times continuously differentiable, non decreasing and concave in W, with $u'(\cdot)>0$ and $u''(\cdot)<0$.

Now, assume we have interior solutions, the first order conditions associated to the above program are:

$$\int_0^1\int_0^1 (\tilde{y}-x_0)u'(W(a_i, \tilde{y}, \tilde{z}))d^2H(y, z)=0, \tag{3}$$

$$\int_0^1\int_0^1 (\tilde{z}-x_0)u'(W(a_i, \tilde{y}, \tilde{z}))d^2H(y, z)=0, \tag{4}$$

In the case of independence, condition (3) and (4) becomes, respectively

$$\int_0^1\int_0^1 (\tilde{y}-x_0)u'(W(a_i, \tilde{y}, \tilde{z}))dF(y|z)dG(z)=\int_0^1\int_0^1 (\tilde{y}-x_0)u'(W(a_i, \tilde{y}, \tilde{z}))dF(y)dG(z) \tag{5}$$

and

$$\int_0^1\int_0^1 (\tilde{z}-x_0)u'(W(a_i, \tilde{y}, \tilde{z}))dF(z|y)dF(y)=\int_0^1\int_0^1 (\tilde{z}-x_0)u'(W(a_i, \tilde{y}, \tilde{z}))dF(y)dG(z) \tag{6}$$

Evaluated at $a_1=0$, and $a_2=0$ (5) and (6) can be written, respectively as

$$(E(\tilde{y})-x_0)\int_0^1 u'(1+x_0+a_2(\tilde{z}-x_0))dG(z) \tag{7}$$

$$(E(\tilde{z})-x_0)\int_0^1 u'(1+x_0+a_1(\tilde{y}-x_0))dF(y) \tag{8}$$

Which have the sign of the expected excess return $(E(\tilde{x})-x_0)$. It follows that a_i is positive if and only if $(E(\tilde{x})-x_0)$ is also positive, that is if $\tilde{x}$ offers a positive risk premium.

3. Effects of wealth on optimal portfolio

We investigate the effect on the demand for risky asset when there are changes in wealth. We design effects of wealth on optimal portfolio by deriving the first order conditions with respect to the portfolio share of risky asset a_i and the agent's end of period wealth W. We first have the next result

Proposition 1. Define the function

$$\Phi(z)\equiv\int_0^1 u'(W(a_i, y, z))(y-x_0)dF(y|z) \;\; \forall z \geq x_0 \tag{9}$$

1) If $R'_a \geq 0$ and $\Phi(\cdot) \geq 0$, then

$$\frac{da_i^*}{dW\left(a_i^*, \tilde{y}, \tilde{z}\right)} \leq 0 \text{, for } i=1, 2 \tag{10}$$

2) If $R'_a < 0$ and $\Phi(\cdot) < 0$, then

$$\frac{da_i^*}{dW\left(a_i^*, \tilde{y}, \tilde{z}\right)} > 0 \text{, for } i=1, 2 \tag{11}$$

where R'_a is the derivative of the coefficient of absolute risk aversion

Proof. Assume we have interior solutions, since u is concave, the first order conditions associate to the above program (P) gives the optimal allocation a_i^* satisfying the following equations:

$$\int_0^1\int_0^1 (\tilde{y} - x_0) u'\left(W\left(a_i^*, \tilde{y}, \tilde{z}\right)\right) d^2 H(y, z) = 0, \tag{12}$$

$$\int_0^1\int_0^1 (\tilde{z} - x_0) u'\left(W\left(a_i^*, \tilde{y}, \tilde{z}\right)\right) d^2 H(y, z) = 0, \tag{13}$$

First, differentiating (11) and (12) with respect to a_1 and a_2, respectively, second since u is concave we have

$$\int_0^1\int_0^1 (\tilde{y} - x_0)^2 u''\left(W\left(a_i^*, \tilde{y}, \tilde{z}\right)\right) d^2 H(y, z) < 0, \tag{14}$$

$$\int_0^1\int_0^1 (\tilde{z} - x_0)^2 u''\left(W\left(a_i^*, \tilde{y}, \tilde{z}\right)\right) d^2 H(y, z) < 0, \tag{15}$$

Defines a_i^* as an implicit function of $W\left(a_i^*, \tilde{y}, \tilde{z}\right)$. We thus obtain

$$\frac{da_1^*}{dW\left(a_i^*, \tilde{y}, \tilde{z}\right)} = -\frac{\int_0^1\int_0^1 (\tilde{y} - x_0) u''\left(W\left(a_i^*, \tilde{y}, \tilde{z}\right)\right) d^2 H(y, z)}{\int_0^1\int_0^1 (\tilde{y} - x_0)^2 u''\left(W\left(a_i^*, \tilde{y}, \tilde{z}\right)\right) d^2 H(y, z)} \tag{16}$$

and

$$\frac{da_2^*}{dW\left(a_i^*, \tilde{y}, \tilde{z}\right)} = -\frac{\int_0^1\int_0^1 (\tilde{z} - x_0) u''\left(W\left(a_i^*, \tilde{y}, \tilde{z}\right)\right) d^2 H(y, z)}{\int_0^1\int_0^1 (\tilde{z} - x_0)^2 u''\left(W\left(a_i^*, \tilde{y}, \tilde{z}\right)\right) d^2 H(y, z)} \tag{17}$$

The denominator is unambiguously negative and clearly the sign of (16) and (17) depends upon the sign of the numerator. Now, starting from (16)

$$-\int_0^1\int_0^1 (\tilde{y} - x_0) u''\left(W\left(a_i^*, \tilde{y}, \tilde{z}\right)\right) d^2 H(y, z) = \int_0^1 y(z) dG(z) \tag{18}$$

where

$$y(z) \equiv -\int_0^1 (\tilde{y} - x_0) u''\left(W\left(a_i^*, \tilde{y}, \tilde{z}\right)\right) dF(y|z) \equiv B \tag{19}$$

Note that $y(z)$ is nonnegative if $B \geq 0$

$$B = -\int_0^1 (\tilde{y} - x_0) u''\left(W\left(a_i^*, \tilde{y}, \tilde{z}\right)\right) dF(y|z) = -\int_0^1 \frac{u''\left(W\left(a_i^*, \tilde{y}, \tilde{z}\right)\right)}{u'\left(W\left(a_i^*, \tilde{y}, \tilde{z}\right)\right)} u'\left(W\left(a_i^*, \tilde{y}, \tilde{z}\right)\right)(\tilde{y} - x_0) dF(y|z)$$

$$= \int_0^1 R_a\left(W\left(a_i^*, \tilde{y}, \tilde{z}\right)\right) u'\left(W\left(a_i^*, \tilde{y}, \tilde{z}\right)\right)\left(\tilde{y} - x_0\right) dF\left(y|z\right) \qquad (20)$$

$$= - \int_0^{x_0} R_a\left(W\left(a_i^*, \tilde{y}, \tilde{z}\right)\right) u'\left(W\left(a_i^*, \tilde{y}, \tilde{z}\right)\right)\left(x_0 - \tilde{y}\right) dF\left(y|z\right) \qquad (21)$$

$$+ \int_{x_0}^1 R_a\left(W\left(a_i^*, \tilde{y}, \tilde{z}\right)\right) u'\left(W\left(a_i^*, \tilde{y}, \tilde{z}\right)\right)\left(\tilde{y} - x_0\right) dF\left(y|z\right)$$

$\forall\ z \in \left[x_0, 1\right]$ and $\forall\ y \in \left[0,\ x_0\right]$ we have
$\tilde{W}\left(a_i^*, \tilde{y}, \tilde{z}\right) = 1 + x_0 + a_1^*\left(\tilde{y} - x_0\right) + a_2^*\left(\tilde{z} - x_0\right) \le 1 + x_0 + a_2^*\left(\tilde{z} - x_0\right) = \tilde{W}\left(a_i^*, \tilde{z}\right)$. Hence
$R'_a > 0$ implies that $R_a\left(W\left(a_i^*, \tilde{y}, \tilde{z}\right)\right) < R_a\left(W\left(a_i^*, \tilde{z}\right)\right)$. Thus

$$- \int_0^{x_0} R_a\left(W\left(a_i^*, \tilde{y}, \tilde{z}\right)\right) u'\left(W\left(a_i^*, \tilde{y}, \tilde{z}\right)\right)\left(x_0 - \tilde{y}\right) dF\left(y|z\right) >$$

$$- \int_0^{x_0} R_a\left(W\left(a_i^*, \tilde{z}\right)\right) u'\left(W\left(a_i^*, \tilde{y}, \tilde{z}\right)\right)\left(\tilde{y} - x_0\right) dF\left(y|z\right) \qquad (22)$$

and $\forall\ z \in \left[x_0, 1\right]$ and $\forall\ y \in \left[x_0, 1\right]$ we have
$\tilde{W}\left(a_i^*, \tilde{y}, \tilde{z}\right) = 1 + x_0 + a_1^*\left(\tilde{y} - x_0\right) + a_2^*\left(\tilde{z} - x_0\right) \ge 1 + x_0 + a_2^*\left(\tilde{z} - x_0\right) = \tilde{W}\left(a_i^*, \tilde{z}\right)$. Hence
$R'_a > 0$ implies that $R_a\left(W\left(a_i^*, \tilde{y}, \tilde{z}\right)\right) > R_a\left(W\left(a_i^*, \tilde{z}\right)\right)$. Thus

$$\int_{x_0}^1 R_a\left(W\left(a_i^*, \tilde{y}, \tilde{z}\right)\right) u'\left(W\left(a_i^*, \tilde{y}, \tilde{z}\right)\right)\left(x_0 - \tilde{y}\right) dF\left(y|z\right) >$$

$$\int_{x_0}^1 R_a\left(W\left(a_i^*, \tilde{z}\right)\right) u'\left(W\left(a_i^*, \tilde{y}, \tilde{z}\right)\right)\left(\tilde{y} - x_0\right) dF\left(y|z\right) \qquad (23)$$

Hence

$$B > \int_0^1 R_a\left(W\left(a_i^*, \tilde{z}\right)\right) u'\left(W\left(a_i^*, \tilde{y}, \tilde{z}\right)\right)\left(\tilde{y} - x_0\right) dF\left(y|z\right) > 0 \qquad (24)$$

since

$$\Phi(z) \equiv \int_0^1 u'\left(W\left(a_i, y, z\right)\right)\left(y - x_0\right) dF\left(y|z\right) > 0$$

Similarly $B = 0$ if $R_a^{'} = 0$, and if $R_a^{'} < 0$ we have that

$$B < \int_0^1 R_a\left(W\left(a_i^*, \tilde{z}\right)\right) u'\left(W\left(a_i^*, \tilde{y}, \tilde{z}\right)\right)\left(\tilde{y} - x_0\right) dF\left(y|z\right) < R_a\left(W\left(a_i^*, \tilde{z}\right)\right)\Phi(z) < 0 \qquad (25)$$

Hence $B < 0$ $\blacksquare$

Analysing the results above, we shall describe the behaviour of the portfolio in term of the behaviour of a simple characteristic of the utility function.

Proposition 1 reveals that, there are many alternative statistics by which portfolio might be characterized. We focus on one and we can assert that, if there are two risky assets and money, then the wealth elasticity of the demand for one risky asset is greater than, equal to, or less than unity as absolute risk aversion is an increasing, constant or decreasing function of wealth.

The intuition of proposition 1 is as follow. Optimal demand of one risky asset depends upon the agent's attitude in the presence of risk.

4. Shifts in returns distribution of one risky asset

We design shifts in one distribution, we consider changes in a_1^* when $\tilde{y}$ undergoes an FDS shift from F^0 to F^1. And we investigate whether or not a first degree dominance shift in one risky asset will result in a higher demand for this risky asset.

For the case of independent $\tilde{y}$ and $\tilde{z}$, following Hadar and Seo, we carry out the comparative static analysis by replacing $H^0(\tilde{y}, \tilde{z}) = F^0(\tilde{y}) \cdot G(\tilde{z})$ with $H^1(\tilde{y}, \tilde{z}) = F^1(\tilde{y}) \cdot G(\tilde{z})$. That is, we assume that $\tilde{y}$ and $\tilde{z}$ are independently distributed both before and after the change in $\tilde{y}$ occurs. That implies that the same change occurs for all conditional CDFs for $\tilde{y}$; that is, $\left[F^1(y|z) - F^0(y|z)\right]$ is the same for each $\tilde{z}$. For this independent case, it is also true that the marginal and conditional CDFs for $\tilde{z}$, $G(\tilde{z})$ and $G(z|y)$, remain unchanged as $\tilde{y}$ is altered.

FDS improvement in the CDF for $\tilde{y}$ are represented by $F^1(y) \geq F^0(y)$ for all y in $[0, 1]$.

Proposition 2. Assume first that the utility function u satisfies $u' > 0$, $u'' < 0$, second $\tilde{y}^j$ and $\tilde{z}$ are stochastically independent, $j = 0, 1$; third, $\int_0^1\int_0^1 u(W(a_i, \tilde{y}, \tilde{z}))d^2H(y, z)$ is maximized at a_i^j. Then $a_i^1 \geq a_i^0$ for any $\tilde{y}^1 \, FSD \, \tilde{y}^0$ if and only if $u'(W(a_i, y, z))(X - x_0)$ ($X = \tilde{y}$ *or* $X = \tilde{z}$) is nondecreasing.

Proof.

Let $x(a_i)$ denote the difference between the derivatives with respect to a_i of expected utility under the two distribution of X; that is,

$$x(a_i) = \int_0^1\int_0^1 (\tilde{y} - x_0)u'(W(a_i, \tilde{y}, \tilde{z}))d\left(F^1(y) - F^0(y)\right)dG(z) \; , \tag{26}$$

the assumptions of the proposition in conjunction of the basic FDS theorem (Fishburn and Vickson, 1978) imply that, $\int_0^1 (\tilde{y} - x_0)u'(W(a_i, \tilde{y}, \tilde{z}))d\left(F^1(y) - F^0(y)\right)dG(z) \geq 0$ for all a_i and $\tilde{z}$, so that $x(a_i) \geq 0$ for all a_i. Evaluating $x(a_i)$ at $a_i = \left(a_i^0\right)^*$ we have

$$x\left(\left(a_i^0\right)^*\right) = \int_0^1\int_0^1 (\tilde{y} - x_0)u'(W(a_i, \tilde{y}, \tilde{z}))d\left(F^1(y)\right)dG(z)\Big|_{a_i = \left(a_i^0\right)^*} \geq 0 \tag{27}$$

since the firs order condition gives

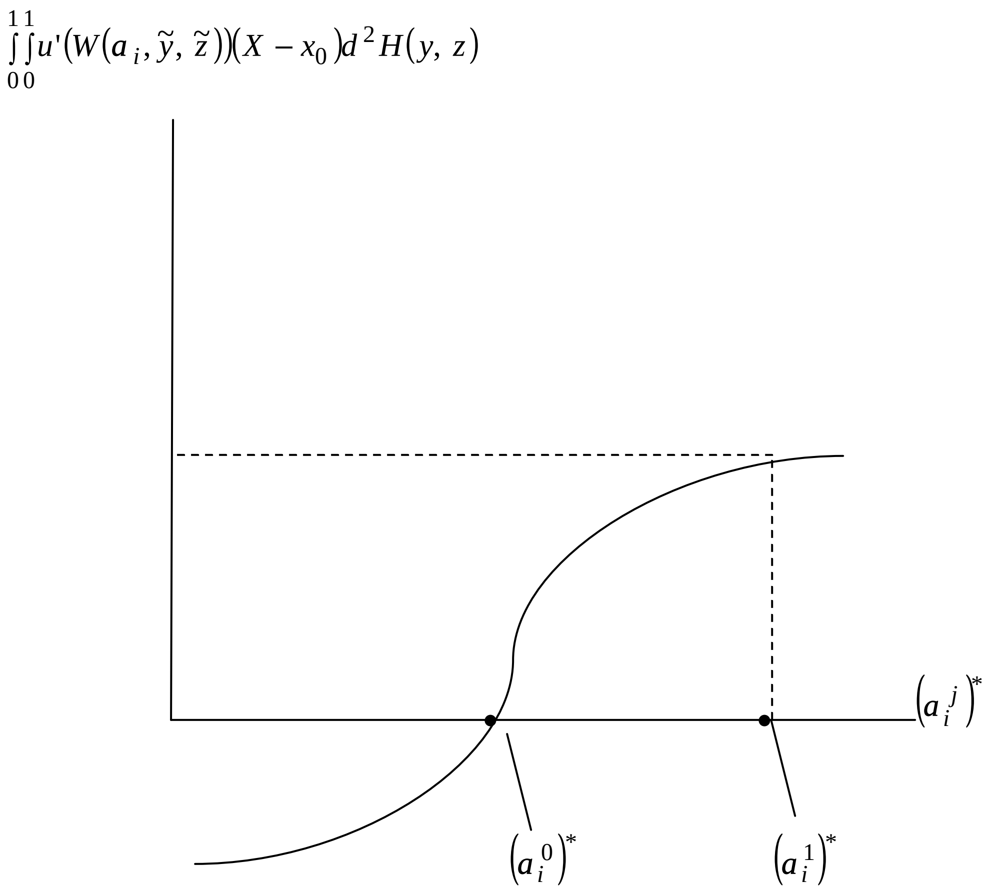
$$\int_0^1\int_0^1 (\tilde{y} - x_0)u'\left(W\left(\left(a_i^0\right)^*, \tilde{y}, \tilde{z}\right)\right)d\left(F^0(y)\right)dG(z) = 0 \tag{28}$$

The same comment applies for

$$x\left(\left(a_i^0\right)^*\right)=\int_0^1\int_0^1(\tilde{z}-x_0)u'(W(a_i,\tilde{y},\tilde{z}))d\left(F^1(y)\right)dG(z)\Big|_{a_i=\left(a_i^0\right)^*}\geq 0, \qquad (29)$$

then the concavity of u implies $a_i^1 \geq a_i^0$ ■

Figure 1: represents effects of FSD on optimal portfolio with one risk-free asset and two risky assets.



It is worth noting that the conditions on utility in proposition 2 are similar as those found in Fishburn and Porter (1976) for FSD changes, and Rothschild and Stiglitz (1971). Thus, with independence, extension to portfolio with one risk free asset and two risky assets preserves the findings from the one risky and one riskless asset case.

Conclusion

The question of whether risk-averse agents should or should not choose diversified portfolio (as opposed to specialized ones) has been analysed quite extensively in literature. In this paper, we have addressed the question of the optimal proportion of one risky asset in a diversified portfolio. For simplicity, the analysis in this paper is confined to random variables whose distributions are confined on the unit interval. Agents are assumed to maximize the expected utility of terminal wealth which is the end-of-period value of portfolio. This paper is concerned with conditions under which the proportion of a given risky asset in the optimal portfolio of a risk averse agent is at least as large as some given proportion.

We then in section 3 looked at portfolio in which the optimal demand of one risky asset depends upon the decreasing, constant or increasing risk aversion. This first result is consistent with those obtained in the portfolio with one riskless and one risky asset. This analysis is facilitated by the use of a special subset of risk-averter. The object of our section 4 is to provide briefly conditions that are sufficient for a risk averter to invest more in the dominating asset. Note that our study could be interpreted in terms of mutual fund separation.

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